Thinking about Negation

When we look at Gentzen's natural deduction rules for intuitionist propositional logic, two oddities stand out: that the *falsum* constant has no introduction rule, and that the introduction and elimination rules for negation are *impure*, both employing the *falsum* constant.

Additionally, the rules for $\wedge$, $\vee$, $\rightarrow$, and $\perp$ are peculiarly *direct*: the connective in question occurs essentially once only, as main connective in the conclusion of the introduction rule and as main connective in the major premise of the elimination rule.

Why is negation different and what are we to learn from this?

Before addressing that question, I want to consider $\perp$ more closely. Why does it not have an introduction rule? In a standard logical language with connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$ and $\perp$ we can restrict the $\perp$-elimination rule to atomic formulas in the conclusion. With that in place, it seems obvious what an introduction rule for $\perp$ should look like (cf. Dummett). But that is a mistake. Rather, that there is no introudction rule tells us something about how $\perp$ is to be understood (and, more generally, about how schematic rules are to be understood).

Does $\perp$ express a proposition? We can try treating $\perp$ less as a proposition and more as a sign of a deductive "dead-end" (cf. Tennant). We then find that treating $\perp$ as a proposition makes for a smoother proof-theory. But which proposition? – The intuitionist asnwer '0 = 1' is not entirely adequate.

How much of classical propositional logic can we obtain employing direct rules? As David Makinson shows in a forthcoming paper, the answer is: exactly intuitionist propositional logic *provided that we treat* $\neg\varphi$ *as, strictly, an abbreviation for* $\varphi \rightarrow \perp$.

What if, like Gentzen, we don't treat negation as an abbreviation? We find that there is no direct intoduction rule for negation. This helps explain why there can be no conclusion of the form $\neg\varphi$ to a derivation that does not employ a *negative* formula as an assumption (possibly discharged in the course of the derivation).

This sits very badly with the proof-theoretic semanticist's conception of grasp of the meaning of a connective being given by knowledge of its introduction and elimination rules.

This suggests wew should rethink the role of introduction rules. Proof-theoretic semantics has been too tied to a verificationist conception of meaning (to which direct introduction rules answer very well). We need to think more carefully about what introduction rules do in proofs from assumptions.

When we attend to the function of introduction rules in proofs, we are lead to a quite different conception of their role, and hence of the form they may take. This allows us to arrive at a formulation of *classical* propositional logic in natural deduction with the subformula property.